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ON THE PURSUIT PROBLEM IN NONLINEAR DIFFERENTIAL GAMES

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We prove a sufficient condition for the termination of pursuit in nonlinear games. We indicate a class of games on a plane, for which this condition is satisfied, we introduce the notion of relative optimality, and we consider an example.

1. Let the motion of a vector z in an n-dimensional Euclidean space R_n be described by the vector differential equation

$$\dot{z} = f(z^{\cdot}, u, v), \qquad u \in P, \qquad v \in Q \tag{1.1}$$

Here the function $f(z, u, v)$ is defined and is continuous for all z, u, v; P and Q are arbitrary compact subsets of the p- and q-dimensional Euclidean spaces R_p and R_q, respectively. The control parameter u corresponds to the pursuing (chasing) object and v to the pursued (escaping) object. Further, a certain terminal set M is specified in R_n. The game consists of the following: the pursuing object tries to lead out the point z onto M, while the pursued object, generally speaking, hinders this. The game is considered terminated when point z falls onto M. All this describes a differential pursuit game (cf. [1]).

Let the game start from a point $z_0 \overline{\in} M$ at $t = 0$. We say that the pursuit from point z_0 can be terminated in a finite time if there exists a number $t(z_0) > 0$ such that under an arbitrary measurable variation $v(t)$ of parameter v we can select a measurable variation $u(t)$ of parameter u such that the solution $z(t)$ of the equation

$$z^{\cdot} = f(z, u(t), v(t)), \qquad z(0) = z_0 \tag{1.2}$$

falls onto M in a time not exceeding the number $t(z_0)$; here, for finding the value $u(t)$ of parameter u at each instant $t \geqslant 0$ we use only the current information: the values $z(t)$ and $v(t)$ of vector z and of parameter v at this same instant t. In what follows we need a generalization of Filippov's lemma [2, 3]. We present it in the necessary form.

Filippov's lemma. If $\varphi(t, u)$ is a continuous n-vector-valued function of the arguments $t \in [\alpha, \beta]$, $u = (u_1, u_2, \ldots, u_r) \in \Pi$, Π is a compactum in an r-dimensional Euclidean space, $y(t)$ is a measurable n-vector-valued function defined on the interval $[\alpha, \beta]$ and $\varphi(t, \Pi) \ni y(t)$, then there exists a measurable function $u(t)$, $\alpha \leqslant t \leqslant \beta$, for which $\varphi(t, u(t)) = y(t)$ for almost all $t \in [\alpha, \beta]$, i.e. the equation $\varphi(t, u) = y(t)$ has a measurable solution.

Let us state a generalization of this lemma.

Lemma 1. If $\psi(t, u, v)$ is a continuous n-vector-valued function of the arguments $t \in [\alpha, \beta]$, $u \in \Pi_1$, $v \in \Pi_2$, Π_1 and Π_2 are compacta in s- and r-dimensional Euclidean spaces, respectively, $v_0(t)$, $y(t)$ are measurable functions defined on $[\alpha, \beta]$ and $\psi(t, \Pi_1, v_0(t)) \ni y(t)$, then the equation $\psi(t, u, v_0(t)) = y(t)$

has a measurable solution.

Proof. For each $t \in [\alpha, \beta]$, by $u_0(t)$ we denote the solution, smallest in the lexicographic sence, of the equation $\psi(t, u, v_0(t)) = y(t)$ [2, 3]. By Luzin's theorem, for any $\varepsilon > 0$ we can find a compact set $\sigma \subset [\alpha, \beta]$, $\beta - \alpha - \text{mes}\, \sigma < \varepsilon$, on which the functions $v_0(t)$, $y(t)$ are continuous. By arguing just the same way as in [2, 3], we can show the measurability of $u_0(t)$ on σ. Because the number ε is arbitrary, the function $u_0(t)$ is also measurable on $[\alpha, \beta]$.

Theorem 1. Let the game be started from a point $z_0 \overline{\in} M$ at $t = 0$. If there exists an absolutely continuous function $\xi(t)$, $0 \leqslant t \leqslant T(z_0)$, for which: (1) $\xi(0) = z_0$, $\xi(\tau_0) \in M$, $\tau_0 = T(z_0)$, (2) $\xi^{\cdot}(t) \in f(\xi(t), P, v)$ for any $v \in Q$ for almost every $t \in [0, \tau_0]$, then we can terminate the pursuit in time $T(z_0)$.

Proof. 1°. From Condition (2) of the theorem it follows that $\xi^{\cdot}(t) \in f(\xi(t), P, Q)$ for almost every $t \in [0, \tau_0]$. We denote the Cartesian direct product $P \times Q$ by Π and the function $f(\xi(t), u, v)$ by $\varphi(t, w)$, where $w = (u, v)$. Obviously, the function $\varphi(t, w)$ is continuous in t, w and the set Π is compact in $R_p \times R_q$. Consequently, all the conditions of Filippov's lemma are satisfied. Therefore, there exists a measurable function $w_0(t)$, defined on the interval $[0, \tau_0]$, for which

$$\varphi(t, w_0(t)) = \xi^{\cdot}(t)$$

Obviously, the components $u_0(t)$, $v_0(t)$ of the measurable function $w_0(t)$ also are measurable and $f(\xi(t), u_0(t), v_0(t)) = \xi^{\cdot}(t)$ for almost all $t \in [0, \tau_0]$. Hence, the function $\xi(t)$ is a solution of Eq. (1.1) (with $u = u_0(t)$, $v = v_0(t)$).

2°. Now let $v = v_1(t)$, $0 \leqslant t \leqslant \tau_0$, be an arbitrary measurable function with values from Q. We denote the function $f(\xi(t), u, v)$ by $\psi(t, u, v)$. The function $\psi(t, u, v)$ is defined for all $t \in [0, \tau_0]$, $u \in P$, $v \in Q$, is continuous in t, u, v, and $\psi(t, P, v_1(t)) \in \xi^{\cdot}(t)$ by virtue of Condition (2) of the theorem. Hence, all the hypotheses of Lemma 1 are satisfied. Therefore, there exists a measurable function $u_1(t)$, $0 \leqslant t \leqslant \tau_0$, for which

$$f(\xi(t), u_1(t), v_1(t)) \equiv \psi(t, u_1(t), v_1(t)) = \xi^{\cdot}(t) \tag{1.3}$$

for almost every t. From (1.3) we see that the absolutely continuous function $\xi(t)$ is a solution of Eq. (1.1) (with $u = u_1(t)$, $v = v_1(t)$).

3°. Suppose that the pursued object chose an arbitrary measurable control $v = v(t)$ whose value at every instant $t \geqslant 0$ becomes known to the pursuer. Then, from the value $v(t)$ he chooses the value $u(t)$ of his own control parameter u at this same instant t so that

$$f(\xi(t), u(t), v(t)) = \xi^{\cdot}(t)$$

Obviously, the solution $z(t)$ of Eq. (1.1), corresponding to the controls $u(t)$, $v(t)$, coincides with $\xi(t)$: $z(t) \equiv \xi(t)$ (see Sect. 2). Therefore, $z(0) = z_0$ and $\xi(\tau_0) = z(\tau_0) \in M$. The theorem is proved.

2. Let us consider nonlinear games on a plane. We indicate conditions under which the game can be completed from the points of a certain region. Further, we prove the optimality of the pursuit time relative to the region (see below for the definition).

Let the motion of vector z be described by the system

$$z_1^{\cdot} = z_2, \qquad z_2^{\cdot} = g(z, u, v) \tag{2.1}$$

Here u, v are scalar control parameters whose range of variation is $P = Q = [-1, 1]$. The terminal set $M = \{0\}$. Concerning the function $g(z, u, v)$ we assume that it is continuous in all arguments for all z and for $u \in P$, $v \in Q$, is continuously differentiable in z_1, z_2 for $u = v = 1$, $u = v = -1$ and for all z. We assume further the fulfillment of the following conditions:

1) No trajectory whatsoever of system (2.1) can go to infinity or come out from infinity within a finite time interval.

2) Let $f_1 = f_1(z) \equiv g(z, 1, 1)$, $f_2 = f_2(z) \equiv g(z, -1, -1)$. For all z and for $i = 1, 2$,

a) $\frac{\partial f_i}{\partial z_1} < -\frac{1}{4}\left(\frac{\partial f_i}{\partial z_2}\right)^2$

b) $\left(\frac{\partial^2 f_i}{\partial z_1, \partial z_2}\right)^2 \leqslant \frac{\partial^2 f_i}{\partial z_1^2} \frac{\partial^2 f_i}{\partial z_2^2}, \qquad (-1)^i \left[\frac{\partial^2 f_i}{\partial z_1^2} - \frac{\partial^2 f_i}{\partial z_2^2}\right] \leqslant 0$

3) $f_1(z) > f_2(z)$ for all z.

4) $f_1(0) > 0 > f_2(0)$

5) For each fixed v the function $g(z, u, v)$ reaches its maximum for $u = 1$ and minimum for $u = -1$. Furthermore, $g(z, 1, v) \geqslant g(z, 1, 1)$, $g(z, -1, v) \leqslant g(z, -1, -1)$.

Let us consider a controlled object described by the system

$$\dot z_1 = z_2, \qquad \dot z_2 = f(z, w) \tag{2.2}$$
$$f(z, w) = {}^1/_2\,[(1 + w) f_1(z) + (1 - w) f_2(z)]$$

Here the control parameter w can take values from the segment $W = [-1, 1]$. For system (2.2) we consider the time-optimal problem of hitting on the origin of the plane R_2. All the hypotheses of Theorem 3.32 of [4] are satisfied. In fact, by virtue of assumption (5) the set $g(z, P, Q) \supset f(z, W)$, i.e. any trajectory of system (2.2) serves simultaneously as a trajectory of system (2.1); therefore, Condition A of Theorem 3.32 is satisfied. Since $f(z, 1) = f_1(z)$, $f(z, -1) = f_2(z)$, Conditions C, D also are satisfied (see 2)). Further, $\partial f / \partial w = f_1 - f_2 > 0$ according to (3) and $f(0, 1) = f_1(0) > 0$, $f(0, -1) = f_2(0) < 0$, according to (4); hence conditions (3.73), (3.74) of [4] also are satisfied. Consequently, when Conditions (1) – (5) are satisfied, a region G ($\subset R_2$) exists for the controlled object (2.2), from any point of which a motion to the origin is possible, which is optimal in region G. The synthesis of controls optimal in region G can be effected in the following manner. The switching line Λ consists of arcs σ_n^-, σ_n^+, $n = 1, 2, \ldots$, while the synthesizing function $w(z)$ equals 1 below line Λ and on arc σ_1^+ and equals -1 above line Λ and on arc σ_1^-.

Theorem 2. Let z_0 be an arbitrary point of region G, $T(z_0)$ be the time in which the phase point goes from z_0 to the origin along an optimal trajectory of system (2.2). Then pursuit from point z_0 can be completed in time $T(z_0)$.

Proof. By $z_0(t)$ we denote the optimal trajectory of system (2.2), connecting point z_0 and the origin. System (2.2) is autonomous; therefore, we can take it that $z_0(0) = z_0$. Then $z_0(T(z_0)) = 0$. Let us convince ourselves that the trajectory $z_0(t)$, $0 \leqslant t \leqslant T(z_0)$ satisfies the hypotheses of Theorem 1. Obviously, Condition (1) is satisfied. Since $\dot z_{01}(t) = z_{02}(t)$, $\dot z_{02}(t) = f(z_0(t), w_0(t))$, where $w_0(t)$, $0 \leqslant t \leqslant T(z_0)$

is the optimal control leading the phase point from z_0 to the origin along trajectory $z_0(t)$. to verify Condition (2) it is sufficient to show that $f(z_0(t), w_0(t)) \subset g(z_0(t), P, Q)$ for any v and for almost every t. We have

$$f(z_0(t), w_0(t)) \subset [g(z_0(t), -1, -1), g(z_0(t), 1, 1)]$$

On the other hand

$$g(z_0(t), P, Q) = [g(z_0(t), -1, v), \; g(z_0(t), 1, v)] \supset [g(z_0(t), -1, -1), g(z_0(t), 1, 1)]$$

Hence $f(z_0(t), w_0(t)) \subset g(z_0(t), P, v)$ for any v and for almost every t. Consequently, by virtue of Theorem 1 the pursuit from z_0 can be terminated.

The theorem from [4] cited above establishes the optimality of the trajectories only in region G, i.e. they are optimal in comparison only with trajectories wholly located in G. Therefore, in the differential game described by system (2.1) we can consider the optimality of the pursuit time relative to region G. We introduce the precise definition.

Definition. Let D be some subset of R_2, containing point z_0. The number $t(z_0)$ is called the optimal pursuit time relative to D if: (1) the pursuit from point z_0 can be completed in time $t(z_0)$, (2) there exists a measurable control $v(t)$, $0 \leqslant t \leqslant t(z_0)$, such that for any measurable control $u(t)$, $0 \leqslant t \leqslant t(z_0)$, the solution $z(t)$, $0 \leqslant t \leqslant t(z_0)$ of system (2.1), corresponding to the controls $u(t)$, $v(t)$ and emerging from z_0 at $t = 0$, satisfies the conditions $z(t) \in D$ for all $t \in [0, t(z_0)]$ and $z(t) \neq 0$ for any $t \in [0, t(z_0))$. Obviously, if $D = R_2$, then optimality as introduced above coincides with optimality in Pontriagin's sense [1].

Theorem 3. If Conditions (1)–(5) are satisfied, then the time $T(z_0)$ is optimal relative to G for any point $z_0 \in G$.

Proof. The possibility of completing the pursuit from an arbitrary point $z_0 \in G$ in time $T(z_0)$ was established in Theorem 2. It remains to prove the validity of the second part of the definition. Assume that the pursued object applies the control $v(t) = w_0(t)$, $0 \leqslant t \leqslant T(z_0)$, while the pursuing object applies an arbitrary control $u(t)$, $0 \leqslant t \leqslant T(z_0)$. The trajectory $z(t)$, $0 \leqslant t \leqslant T(z_0)$, corresponding to $u(t)$, $v(t)$ connects the points z_0, $z(T(z_0))$ and is located wholly in G (see the definition). To be specific let z_0 be above Λ, for $0 \leqslant t < t_1$ let the trajectory $z(t)$ lie in a two-dimensional cell Σ_1, let it be a part of a one-dimensional ν of second kind for $t_1 \leqslant t < t_2$, let it be a part of a two-dimensional cell Σ_2 for $t_2 \leqslant t < t_3$, etc. [4], and, finally, for $t_k \leqslant t \leqslant T < T(z_0)$ let it hit into the origin on a cell of first kind.

As is known [4], the function $\omega(z) \equiv -T(z)$, $z \in G$, called the Bellman function, is continuously differentiable in the region $G \setminus \Lambda$ and satisfies in it the Bellman equation

$$\max_{w \in W} \left[\frac{\partial \omega(z)}{\partial z_1} z_2 + \frac{\partial \omega(z)}{\partial z_2} f(z, w)\right] = 1 \tag{2.3}$$

$$\frac{\partial \omega(z)}{\partial z_1} z_2 + \frac{\partial \omega(z)}{\partial z_2} f(z, -1) = 1, \quad \text{if } z \text{ is above } \Lambda$$

The function $z(t)$, $0 \leqslant t < t_1$, is absolutely continuous, while the function $\omega(z)$ is smooth in the region $G \setminus \Lambda$. Therefore [5], their superposition $\omega(z(t)), 0 \leqslant t < t_1$ is absolutely continuous. Hence, for almost all $t \in [0, t_1]$ the derivative $d\omega(z(t))/dt$ exists and can be computed by the formula

$$\frac{d\omega(z(t))}{dt} = \frac{\partial\omega(z(t))}{\partial z_1} z_1(t) + \frac{\partial\omega(z(t))}{\partial z_2} g(z(t), u(t), -1) \tag{2.4}$$

Now let $\varepsilon > 0$ be an arbitrary number, $\varepsilon < t_1$. We consider (2.4) for $0 \leqslant t \leqslant t_1 - \varepsilon$. Since the function $\partial\omega(z(t))/\partial z_2 < 0$ [4] for $t \in [0, t_1 - \varepsilon]$, according to (2.3) we have $d\omega(z(t))/dt \leqslant 1$. Hence, $\omega(z(t_1 - \varepsilon)) - \omega(z(0)) \leqslant t_1 - \varepsilon < t_1$. Hence, because ε is arbitrary, we obtain $t_1 \geqslant \omega(z(t_1)) - \omega(z_0)$. Suppose now that $z(t)$ lies in a one-dimensional cell of first kind for $t \in [\tau_1, \tau_2]$. Because of the special form of system (2.1) this is possible if and only if

$$g(z(t), u(t), w_0(t)) = f(z(t), 1), \quad g(z(t), u(t), w_0(t)) = f(z(t), -1)$$

Consequently, the phase point moves along trajectory $z(t)$ at the same velocity with which it moves for system (2.2) along Λ from $z(\tau_1)$ to $z(\tau_2)$. Hence, $\tau_2 - \tau_1 = \omega(z(\tau_2)) - \omega(z(\tau_1))$.

It is known that if $z(t)$ is a part of a two-dimensional or one-dimensional cell of first kind for $\tau < t < s$, then $s - \tau \geqslant \omega(z(s)) - \omega(z(\tau))$. But in the given case the phase point $z(t)$ can move for some time on a one-dimensional cell of first kind. It can be proved that if $z(t) \in \nu$, $t_1 \leqslant t \leqslant t_2$, then $t_2 - t_1 \geqslant \omega(z(t_2)) - \omega(z(t_1))$. To do this it suffices to prove the validity of the Bellman equation on ν, i.e. it is sufficient that [6]: (a) the optimal trajectories of system (2.2) should not only approach (this follows from Conditions (1)-(5)) but also depart from cell ν at a nonzero angle, (b) the level lines of function $\omega(z)$ at points ν do not touch cell ν.

Let us first prove the validity of condition (a). The optimal trajectories of system (2.2), moving on cell Σ_2, approach a certain one-dimensional cell ν_1 at a nonzero angle [4]. Let z° be an arbitrary point of cell ν_1 and $z^\circ(t)$ be an optimal trajectory of system (2.2) passing through it. Let $\varphi(\Delta)$, $|\Delta| < \varepsilon$ be the equation of cell ν_1 in the neighborhood of point z° and let $\varphi(0) = z^\circ$. By $z^\Delta(t)$ we denote an optimal trajectory of system (2.2) passing through point $\varphi(\Delta)$. Because system (2.2) is autonomous, we can take $z^\Delta(0) = \varphi(\Delta)$, $|\Delta| < \varepsilon$. The trajectory $z^\Delta(t)$ intersects cell ν at some $t = \theta(\Delta)$, $\theta(\Delta) < 0$. As was proved in [4], the function $\theta(\Delta)$ depends smoothly on the parameter Δ. By virtue of the smoothness of cell ν_1, the function $\varphi(\Delta)$, $|\Delta| < \varepsilon$ is also smooth. We have $\varphi^\cdot(\Delta) = \varphi(0) + \varphi^\cdot(0)\Delta + o(\Delta)$ (here and further on $o(\Delta)$ denotes an infinitesimal of order higher than the first relative to Δ). But $z^\Delta(0) = \varphi(\Delta)$, $z^\circ(0) = \varphi(0) = z^\circ$. Consequently [4], $z^\Delta(\theta(\Delta)) = z^\circ(\theta(\Delta)) + \delta z(\theta(\Delta))\Delta + o(\Delta)$. Here $\delta z(t)$ denotes the solution of the variational system

$$\delta z_1^\cdot = \delta z_2, \quad \delta z_2^\cdot = \frac{\partial f(z^\circ(t), w_0(t))}{\partial z_1}\delta z_1 + \frac{\partial f(z^\circ(t), w_0(t))}{\partial z_2}\delta z_2 \tag{2.5}$$

with initial condition $\delta z(0) = \varphi^\cdot(0)$. Since it is obvious that $\theta(\Delta) = \theta(0) + \theta'(0)\Delta + o(\Delta)$, then

$$z^\Delta(\theta(\Delta)) = z^\circ(\theta(0)) + [z^{\cdot\circ}(\theta(0))\theta'(0) + \delta z(\theta(0))]\Delta + o(\Delta) \tag{2.6}$$

The point $z^\Delta(\theta(\Delta))$ belongs to cell ν for all $|\Delta| < \varepsilon$. Therefore, by virtue of (2.6) the vector

$$\delta z(\theta(0)) + z^{\cdot\circ}(\theta(0))\theta'(0) \tag{2.7}$$

is tangent to cell ν at the point $z^\circ(\theta(0))$. We now prove that vector (2.7) is not collinear with vector $z^{\cdot\circ}(\theta,(0))$, i.e. the trajectory $z^\circ(t)$ departs from cell ν at a nonzero angle. Assume that $\delta z(\theta(0)) + z^{\cdot\circ}(\theta(0))\,\theta'(0) = \lambda z^{\cdot\circ}(\theta(0))$, $\lambda \neq 0$. It can be checked that the function $z^{\cdot\circ}(t)$, $\theta(0) \leqslant t \leqslant 0$ is a solution of system (2.5). Consequently, the function

$$\delta z(t) + z^{\cdot\circ}(t)\,\theta'(0), \quad \theta(0) \leqslant t \leqslant 0$$

also is a solution of system (2.5). By virtue of the uniqueness theorem

$$\delta z(t) + z^{\cdot\circ}(t)\,\theta'(0) \equiv \lambda z^{\cdot\circ}(t) \tag{2.8}$$

When $t = 0$, from (2.7) we have

$$\delta z(0) + z^{\cdot\circ}(0)\,\theta'(0) = \lambda z^{\cdot\circ}(0) \tag{2.9}$$

But equality (2.9) is possible if and only if the vector $z^{\cdot\circ}(0)$, i.e. the tangent vector to trajectory $z^\circ(t)$ at the instant $t = 0$, is collinear with the vector $\varphi^{\cdot}(0)$, i.e. the tangent vector to cell ν_1 at point z°. We have arrived at a contradiction because, as was noted above, the trajectory $z^\circ(t)$ approaches ν_1 at a nonzero angle. Thus, the trajectory $z^\circ(t)$ departs from ν_1 at a nonzero angle. Condition (a) is proved.

We proceed to the proof of the condition (b). By $\psi(t)$ we denote a solution of the adjoint system (4), corresponding to the optimal trajectory $z^\circ(t)$ and to the control $u_0(t)$. We assume that $z^\circ(t) \in \Sigma_1$, $0 \leqslant t < \tau_1$ and $z^\circ(t) \in \Sigma_2$, $\tau_1 < t < \tau_2$. As is known [4], the vector $\psi(t) = \lambda_1 \operatorname{grad} \omega(z^\circ(t))$, $\lambda_1 > 0$, for $t \in [0, \tau_1)$, and the vector $\psi(t) = \lambda_2 \operatorname{grad} \omega(z^\circ(t))$, $\lambda_2 > 0$, for $t \in (\tau_1, \tau_2)$, i.e. at points of trajectory $z^\circ(t)$, lying in Σ_1, Σ_2, the vector $\psi(t)$ is directed orthogonally to the level line. By virtue of condition (a) the level line of $\omega(z) = \omega(z^\circ(t))$ is smooth at point $z^\circ(\tau_1)$ [6]. Now, from continuity considerations we conclude that $\psi(\tau_1) = \lambda_1 \operatorname{grad} \omega(z^\circ(\tau_1))$. But [4] the second component of vector $\psi(\tau_1)$ equals zero. Therefore, the tangent vector to the level line of $\omega(z) = \omega(z^\circ(\tau_1))$ at point $z^\circ(\tau_1)$ is directed parallely to the z_2-axis. In [4] it was proved that cell ν does not have vertical tangents. Hence, the level line of $\omega(z) = \omega(z^\circ(\tau_1))$ does not touch ν at point $z^\circ(\tau_1)$. Since $z^\circ(\tau_1)$ ranges over the whole cell ν, condition (b) is proved.

We have
$$\begin{aligned} T &= (T - t_k) + (t_k - t_{k-1}) + \ldots + (t_2 - t_1) + (t_1 - 0) \geqslant \\ &[\omega(z(T)) - \omega(z(t_k))] + [\omega(z(t_k)) - \omega(z(t_{k-1}))] + \ldots \\ &+ [\omega(z(t_2)) - \omega(z(t_1))] + [\omega(z(t_1)) - \omega(z(0))] + \ldots \\ &= -\omega(z(0)) = T(z_0) \end{aligned}$$

We have arrived at a contradiction because $T < T(z_0)$ by assumption. The theorem is proved.

3. Example. Let the game be described by the system [4]

$$z^{\cdot}_1 = z_2, \quad z^{\cdot}_2 = -\omega^2 z_1 - 2\delta z_2 + \rho u - \sigma v \tag{3.1}$$

Here ρ is a positive and ω^2, δ, σ are nonnegative numbers, $\rho > \sigma$, $\delta^2 < \omega^2$, the sets $P = Q = [-1,1]$. $M = \{0\}$. Conditions (1)−(5) are easily verified for (3.1). As is known [4], the region G coincides with the whole plane of variables z_1, z_2. Hence, optimality relative to G for (3.1) turns into optimality in Pontriagin's sense.

Note. Example (3.1) relates to the class of linear one-type objects [7]. By using the extremal sighting method we can establish the possibility of completing the pursuit from any point when the pursuer has less information available (at each instant $t \geqslant 0$ he knows only the value $z(t)$ of the phase variable z). As a rule this situation is common in linear differential games [8, 9].

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ON THE ESTIMATION OF CERTAIN PERFORMANCE INDICES IN LINEAR STATIONARY CONTROLLED SYSTEMS

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We consider the behavior of a closed-loop stationary controlled system when the forcing functions belong to a certain class of functions (the Bulgakov problem [1, 2]). We derive estimates for the modulus of the maximum value of the output and for the largest accumulation of system errors.

1. Consider the system of equations

$$c_0 y^{(n)} + c_1 y^{(n-1)} + \ldots + c_{n-2} y'' + y' = k\varepsilon_x(t) \qquad (1.1)$$
$$y^{(n-1)}(0) = \ldots = y(0) = 0$$
$$\varepsilon_x(t) = x(t) - y(t)$$

Equations (1.1) describe the behavior of a closed-loop linear astatic automatic control